Evolution of solitons over a randomly rough seabed

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(Received 31 August 2003; revised manuscript received 15 December 2003; published 1 July 2004)

For long waves propagating over a randomly uneven seabed, we derive a modified Korteweg–de Vries (KdV) equation including new terms representing the effects of disorder on amplitude attenuation and wave phase. Analytical and numerical results are described for the evolution of a soliton entering a semi-infinite region of disorder, and the fission of new solitons after passing over a finite region of disorder.

DOI: 10.1103/PhysRevE.70.016302 PACS number(s): 47.90.+a, 46.65.+g, 47.45.−n, 05.45.Yv

I. INTRODUCTION

Attenuation due to multiple scattering of waves by random irregularities can be of importance in a variety of scientific and technological endeavors (see, e.g. [1]). After the seminal work by Anderson [2] in condensed-matter physics, tools of modern theoretical physics have been applied to linear wave problems in many branches of classical physics, as summarized by Sheng [3,4]. Mathematical treatments of linear waves such as sound and electromagnetic waves can be found in [5–8]. In recent years, nonlinear waves in weakly random media have also received considerable attention in the literature of mathematical physics. Some authors have studied the effects of prescribed random forcing [9]. Many others have treated the effects of a random potential (a term linear or nonlinear in the unknown and/or its derivatives, with a random coefficient) when added to a classical evolution equation (KdV, nonlinear Schrödinger or Sine-Gordon). Extensive surveys of soliton dynamics under random perturbations have been given in [10–12]. Further contributions include the works by Devillard and Souillard [13], Garnier [14,15] and many others.

In the linear theory of classical waves, the main physical consequence of disorder is the spatial attenuation due to multiple scattering, a phenomenon related to Anderson localization [2] in quantum systems. The effect is undoubtedly of importance in many physical contexts, for example, in fiber optics where electromagnetic signals must travel hundreds of kilometers over which random material inhomogeneities must exist. It is equally important in coastal oceanography where bathymetric irregularities contribute to sea wave attenuation by wave radiation, in addition to other dissipative mechanisms by surface-wave breaking, internal and bottom friction, etc. Earliest theories on sea waves over disordered seabed are due to Hasselman [16] and Long [17] who employed the technique of Feynman diagrams. More recent linear theories are represented by [18] and [19]. Considering nonlinearity, Rosales and Papanicolaou [20] examined solitons over relatively short and steep bottom roughness (both height and length are comparable to the typical water depth) and found that the randomness affects the wave speed. With similar assumptions on the length scales, Nachbin and Solna [21] studied wave scattering by bottom disorder. For these steep roughnesses, both forward and reflected waves are comparable; the coupling of their spectral amplitudes has been examined [39]. Further asymptotic results for steep roughness have been discussed recently by Nachbin [22]. In a recent note Fougue *et al.* [23] considered time-reversal of linear dispersive and nonlinear hyperbolic waves due to weak disorder, as two limits of Boussinesq system.

In coastal seas, the bathymetric roughness extending over large areas is often gentle in local slope and small in amplitude. Using the multiple-scales method, Kawahara [24] has derived evolution equations for unidirectional water waves of both long and intermediate wavelength relative to the depth, without however analyzing the physics. Pertinent approximations for certain model wave equations have been treated by Benilov and Pelinofskii [25]. Recently Mei and associates have studied two- and three-dimensional problems for narrow-banded Stokes waves of intermediate wavelength-to-depth ratio. Similar to [24] but different from [19], both the local height and length of the bed roughness are assumed to be comparable respectively to the wave amplitude and wavelength. For unidirectional waves over the one-dimensional random seabed, the effects on sideband instability and the scattering of soliton envelopes have been investigated [26]. For several two-dimensional regions of random bathymetry, the diffraction and attenuation of Stokes waves have been investigated [27]. In particular, dark solitons are found in the wake of an elongated area. For long periodic waves in shallow water, harmonic generation is known to prevail over a smooth bed [28,29]. Assuming the roughness height to be smaller than the mean depth by a factor of order $KH \ll 1$, and the roughness length scale to be comparable to the wavelength, the influence of disorder on harmonic generation has been studied by Grataloup and Mei [30]. It is found that the amplitudes of the fundamental and higher harmonics are governed by coupled nonlinear equations similar to those in optics [31], but with additional terms whose complex coefficients are deterministic and related to certain correlation functions of disorder.

In this paper we examine the transient propagation of a soliton over a shallow water with a weakly random bathymetry. The scale assumptions are the same as those in [24] and Grataloup and Mei [30], but different from [20–22]. After deriving the asymptotic equation, we shall examine, both analytically and numerically, how incoherent multiple scattering affects the phase and the amplitude of the coherent wave at the leading order.

II. ASYMPTOTIC EQUATION FOR UNIDIRECTIONAL WAVES

We begin with the well-known Boussinesq equations for long waves in shallow water. In physical dimensions, the horizontal and vertical coordinate are denoted by x^*, z^*, time by t^*, the depth-averaged velocity by u^*, the free surface displacement by η^*, the constant mean depth by H, and the depth variations by b^*. Let K^{-1} be the characteristic horizontal length and a the characteristic wave amplitude, we assume that $b^*/H = O(KH) \ll 1$, and introduce normalized variables (without primes) as follows:

$$x = Kx^*, \quad z = \frac{z^*}{H}, \quad t = K\sqrt{gH}t^*,$$

$$u = \frac{u^*}{\frac{a}{H}\sqrt{gH}}, \quad \eta = \frac{\eta^*}{a}, \quad b = \frac{b^*}{KH^2}. \tag{1}$$

Under the assumptions

$$\epsilon = \frac{a}{H} \ll 1, \quad \mu = KH \ll 1, \quad \text{and} \quad \epsilon = O(\mu^2), \tag{2}$$

the Boussinesq approximation for mass and momentum conservation are, to the first order in ϵ and μ^2,

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(1 - \mu b + \epsilon \eta)u] = 0 \tag{3}$$

and

$$\frac{\partial u}{\partial t} + \epsilon u \frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{\mu^2}{3}\frac{\partial^3 u}{\partial x^2 \partial t}. \tag{4}$$

We assume that the depth fluctuation $b(x)$ is a stationary and random function of x, with zero mean.

Within the stated accuracy, the preceding two equations can be combined to give the stochastic differential equation,

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = -\mu \frac{\partial}{\partial x}\left(b(x)\frac{\partial \eta}{\partial x}\right) + \epsilon \frac{\partial^2}{\partial x^2}\left(u^2 + \frac{\eta^2}{2}\right) + \frac{\mu^2}{3}\frac{\partial^4 \eta}{\partial x^4}. \tag{5}$$

Our objective is to study the evolution of a soliton after entering a rough bed in the region of $x > 0$. In the region of smooth bed, $x < 0$ where $b = 0$, the incident soliton is known to be described by

$$\eta^*(x,t) = a \operatorname{sech}^2 \frac{\sqrt{3}}{2}\left(\frac{a}{H^3}\right)^{1/2}(x^* - Ct^*),$$

with

$$C = \sqrt{gH}\left(1 + \frac{a}{H}\right). \tag{6}$$

It is convenient to choose the wavelength of the incident soliton as the horizontal scale $1/K$, i.e.,

$$K = \left(\frac{a}{H^3}\right)^{1/2} \tag{7}$$

so that $\epsilon = \mu^2$.

We shall now derive the statistical average of (5). In anticipation that the small disorder affects the leading order after a long distance inversely proportional to the mean square of the disorder, we introduce two space variables x and $X = \mu^2 x$ and expand u and η as power series of μ:

$$\eta(x,X;t) = \eta_0 + \mu \eta_1 + \mu^2 \eta_2 + O(\mu^3), \tag{8}$$

$$u(x,X;t) = u_0 + \mu u_1 + \mu^2 u_2 + O(\mu^3). \tag{9}$$

The perturbation equations are easily found to be

$$\frac{\partial^2 \eta_0}{\partial t^2} - \frac{\partial^2 \eta_0}{\partial x^2} = 0, \tag{10}$$

$$\frac{\partial^2 \eta_1}{\partial t^2} - \frac{\partial^2 \eta_1}{\partial x^2} = -\frac{\partial}{\partial x}\left(b(x)\frac{\partial \eta_0}{\partial x}\right), \tag{11}$$

$$\frac{\partial^2 \eta_2}{\partial t^2} - \frac{\partial^2 \eta_2}{\partial x^2} = -\frac{\partial}{\partial x}\left(b(x)\frac{\partial \eta_1}{\partial x}\right) + 2\frac{\partial^2 \eta_0}{\partial x \partial X} + \frac{\partial^2}{\partial x^2}\left(u_0^2 + \frac{\eta_0^2}{2}\right) + \frac{1}{3}\frac{\partial^4 \eta_0}{\partial x^4}. \tag{12}$$

At the leading order, $O(1)$, the governing wave equation (10) is homogeneous and deterministic; the solution must represent coherent waves. We limit ourselves to a right-going wave with vanishing amplitude at $x - t \sim -\infty$,

$$\eta_0(x,X;t) = u_0(x,X;t) = \zeta(X;\sigma), \tag{13}$$

where $\sigma = x - t$. The explicit dependence on X and σ is yet undetermined.

At the first order $O(\mu)$, the inhomogeneous equation (11) has random forcing, and can be solved by using the following well-known Green's function (e.g., [32])

$$G(x,t;x',t') = \tfrac{1}{2}H[(t - t') - |x - x'|], \tag{14}$$

where $H(z)$ is the Heaviside step function. The formal solution is

$$\eta_1(x,X;t) = -\int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} dx' G(x,t;x',t')\frac{\partial}{\partial x'} \times \left(b(x')\frac{\partial \zeta(x' - t', X')}{\partial x}\right). \tag{15}$$

Clearly η_1 represents the incoherently scattered wave and is random with zero mean.

We now take the ensemble average of (12), and obtain

$$\frac{\partial^2 \langle \eta_2 \rangle}{\partial t^2} - \frac{\partial^2 \langle \eta_2 \rangle}{\partial x^2} = -\frac{\partial}{\partial x}\left\langle b(x)\frac{\partial \eta_1}{\partial x}\right\rangle + 2\frac{\partial^2 \zeta}{\partial x \partial X} + \frac{3}{2}\frac{\partial^2 \zeta^2}{\partial x^2} + \frac{1}{3}\frac{\partial^4 \zeta}{\partial x^4}. \tag{16}$$

By virtue of (13), the last three terms are all functions of σ

$=x-t$, hence are homogeneous solutions of the averaged wave equation. Together they can be transformed to

$$2\frac{\partial^2\zeta}{\partial\sigma\,\partial X}+\frac{3}{2}\frac{\partial^2\zeta^2}{\partial\sigma^2}+\frac{1}{3}\frac{\partial^4\zeta}{\partial\sigma^4}. \tag{17}$$

We now examine the first forcing term by using (15),

$$\left\langle -b(x)\frac{\partial\eta_1}{\partial x}\right\rangle=\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dx'\times\left[\frac{\partial}{\partial x'}\left(\langle b(x)b(x')\rangle\frac{\partial\zeta}{\partial x'}\right)\right]\frac{\partial G(x,x',t,t')}{\partial x}. \tag{18}$$

By assuming the disorder to be homogeneous in the short scale, the autocorrelation of depth fluctuations is

$$\langle b(x)b(x')\rangle\equiv\Gamma(\xi)=\Gamma(x'-x) \tag{19}$$

which will be assumed to be a positive and even function of $\xi=x'-x$. It is shown in Appendix A that

$$\left\langle -b(x)\frac{\partial\eta_1}{\partial x}\right\rangle=\frac{1}{\pi}\int_{-\infty}^{\infty}\beta(k,X)\hat{\zeta}(X;k)e^{ik(x-t)}dk, \tag{20}$$

where $\hat{\zeta}$ is the Fourier transform of ζ,

$$\hat{\zeta}(k,X)=\int_{-\infty}^{\infty}\zeta(x'-t';X)e^{-ik(x'-t')}dx,$$

and $\beta(k,X)$ is the complex coefficient defined by

$$\beta=\frac{ik}{4}\int_{-\infty}^{\infty}\frac{\partial}{\partial\xi}(\Gamma(\xi,X)e^{ik\xi})\mathrm{sgn}(\xi)e^{ik|\xi|}d\xi. \tag{21}$$

In view of (20), we get

$$-\frac{\partial}{\partial x}\left\langle b(x)\frac{\partial\eta_1}{\partial x}\right\rangle=\frac{1}{\pi}\frac{\partial}{\partial\sigma}\int_{-\infty}^{\infty}e^{ik\sigma}\beta(k,X)\hat{\zeta}(k,X)dk. \tag{22}$$

Thus all forcing terms on the right of (16) are function of $\sigma=x-t$. To avoid unbounded resonance for $\langle\eta_2\rangle$, their sum must vanish. After integrating this solvability condition with respect to σ, we obtain the asymptotic equation for the leading order displacement, as seen by an observer traveling at the linear phase speed (being unity in dimensionless form and $\sqrt{gH}$ in physical scale) during a very long course of propagation,

$$\frac{\partial\zeta}{\partial X}+\frac{3}{2}\zeta\frac{\partial\zeta}{\partial\sigma}+\frac{1}{6}\frac{\partial^3\zeta}{\partial\sigma^3}=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\beta(k,X)\hat{\zeta}(k,X)e^{ik\sigma}dk. \tag{23}$$

This is just a KdV equation modified by the additional term on the right-hand side representing the scattering effect of disorder. The timelike coordinate X represents the distance traveled by the wave. Different from many existing theories based on various models of random potentials, this new term is deterministic.

Equation (23) can be brought to a more explicit form as follows. From (21), we obtain by partial integration,

$$\beta=\frac{ik}{4}\{\Gamma(\xi)e^{ik\xi}\,\mathrm{sgn}(\xi)e^{ik|\xi|}\}_{-\infty}^{\infty}-\frac{ik}{4}\int_{-\infty}^{\infty}(\Gamma(\xi)e^{ik\xi})ike^{ik|\xi|}d\xi$$
$$=-ik\frac{\Gamma(0)}{2}+\frac{k^2}{4}\int_{-\infty}^{\infty}\Gamma(\xi)e^{ik\xi}e^{ik|\xi|}d\xi, \tag{24}$$

where

$$\frac{d}{d\xi}(\mathrm{sgn}(\xi)e^{ik|\xi|})=ike^{ik|\xi|} \tag{25}$$

is used. Since the integrand in the last integral is even in ξ, we find

$$\int_{-\infty}^{\infty}\Gamma(\xi)e^{ik\xi}e^{ik|\xi|}d\xi=\frac{1}{2}\int_{-\infty}^{\infty}\Gamma(\xi)d\xi+\int_0^{\infty}\Gamma(\xi)\cos 2k\xi d\xi+i\int_0^{\infty}\Gamma(\xi)\sin 2k\xi d\xi=\frac{1}{2}\hat{\Gamma}(0)+\frac{1}{2}\hat{\Gamma}(2k)+ik\hat{P}(2k), \tag{26}$$

where the third term is obtained after partial integration, with $P(\xi)$ being defined by

$$P(\xi)=\int_{|\xi|}^{\infty}\Gamma(u)du. \tag{27}$$

Hence β in (24) can be rewritten as

$$\beta=-ik\frac{\Gamma(0)}{2}+k^2\left\{\frac{1}{2}\hat{\Gamma}(0)+\frac{1}{2}\hat{\Gamma}(2k)+ik\hat{P}(2k)\right\}. \tag{28}$$

Substituting (28) into (23), we get, after invoking the convolution theorem,

$$\frac{\partial\zeta}{\partial X}+\frac{3}{2}\zeta\frac{\partial\zeta}{\partial\sigma}+\frac{1}{6}\frac{\partial^3\zeta}{\partial\sigma^3}=\frac{\Gamma(0)}{2}\frac{\partial\zeta}{\partial\sigma}+\frac{\hat{\Gamma}(0)}{8}\frac{\partial^2\zeta}{\partial\sigma^2}+\frac{1}{16}\int_{-\infty}^{\infty}\Gamma\left(\frac{\sigma-\sigma'}{2}\right)\frac{\partial^2\zeta}{\partial\sigma'^2}d\sigma'+\frac{1}{8}\int_{-\infty}^{\infty}P\left(\frac{\sigma-\sigma'}{2}\right)\frac{\partial^3\zeta}{\partial\sigma'^3}d\sigma'. \tag{29}$$

This equation can be reduced to the one found earlier in [24]. We find the form here more convenient for physical interpretation and for further approximate analysis. Physically the coherent part of the wave which dominates the leading order is affected by disorder only on the average, through new terms on the right-hand side of the extended KdV equation. In particular, the first term on the right-hand side represents reduction of the phase velocity. Because $\Gamma(\xi)$ is positive-definite, dispersion is also reduced through the fourth term. In addition, the second and third terms are of Burgers' form and signify diffusion, hence would lead to spatial attenuation. These effects of disorder have been noted in [23].

One of the physical implications of the new terms above can be found at once by studying the energy of the coherent wave. Multiplying both sides of (23) and integrating the result with respect to σ, we get

$$\frac{d}{dX}\int_{-\infty}^{\infty}\frac{\zeta^2}{2}d\sigma = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\beta(k)|\hat{\zeta}(k)|^2dk.$$

Since $\zeta(\sigma)$ is real in all σ, its Fourier transform $\hat{\zeta}(k)$ and its complex conjugate must satisfy $\hat{\zeta}(-k)=\hat{\zeta}^*(k)$. By the fact that $\mathrm{Re}(\beta(k))$ is even and $\mathrm{Im}(\beta(k))$ is odd in k, it follows that

$$\frac{d}{dX}\int_{-\infty}^{\infty}\frac{\zeta^2}{2}d\sigma = -\frac{1}{\pi}\int_{0}^{\infty}\mathrm{Re}(\beta(k))|\hat{\zeta}(k)|^2dk. \tag{30}$$

From (28), $\mathrm{Re}(\beta(k))>0$ for all positive k. The right-hand side of (30) is negative. Physically, energy is drained from the coherent wave for the radiation of the much smaller, randomly scattered incoherent waves, leading to spatial attenuation of η_0. We point out that the attenuation of sinusoidal waves of infinitesimal amplitude is characterized by exponential reduction of the transmission waves with the total extent of disorder. It is known that for some nonlinear waves, spatial attenuation can also be algebraic [13], as will be the case here.

III. GAUSSIAN CORRELATION FUNCTION

For better insight we choose the correlation function to be Gaussian, with the correlation length taken to be $l^*=l/K$. The dimensional correlation function is,

$$\langle b^*(x^*)b^*(x^{*\prime})\rangle = D^{*2}\exp\left(\frac{(x^{*\prime}-x^*)^2}{2l^{*2}}\right), \tag{31}$$

where D^* is the root-mean-square amplitude of the random roughness. The normalized correlation function is

$$\Gamma(\xi) = D^2\exp\left(-\frac{\xi^2}{2l^2}\right), \tag{32}$$

where

$$D^2 = \frac{D^{*2}}{H^2}\frac{1}{\mu^2} = \frac{D^{*2}}{aH} = O(1). \tag{33}$$

It follows that

$$\hat{\Gamma}(k) = \sqrt{2\pi}D^2 l\exp\left(-\frac{k^2l^2}{2}\right) \tag{34}$$

and

$$P(\xi) = \int_{|\xi|}^{\infty}\Gamma(u)du = \sqrt{\frac{\pi}{2}}D^2 l\,\mathrm{erfc}\left(\frac{|\xi|}{\sqrt{2}l}\right), \tag{35}$$

where $\mathrm{erfc}(z)$ is the complementary error function.

The Fourier transform of $P(\xi)$ is

$$\hat{P}(k) = \frac{\sqrt{2\pi}D^2l^2}{kl}\exp\left(-\frac{k^2l^2}{2}\right)\mathrm{erfi}\left(\frac{kl}{\sqrt{2}}\right), \tag{36}$$

where $\mathrm{erfi}(z)$ is the imaginary error function defined by

$$\mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{t^2}dt. \tag{37}$$

Consequently (29) becomes

$$\frac{\partial\zeta}{\partial X}+\frac{3}{2}\zeta\frac{\partial\zeta}{\partial\sigma}+\frac{1}{6}\frac{\partial^3\zeta}{\partial\sigma^3} = D^2\left\{\frac{1}{2}\frac{\partial\zeta}{\partial\sigma}+\frac{\sqrt{2\pi}l}{8}\frac{\partial^2\zeta}{\partial\sigma^2}+\frac{1}{16}\int_{-\infty}^{\infty}\exp\left(-\frac{|\sigma-\sigma'|^2}{8l^2}\right)\times\frac{\partial^2\zeta}{\partial\sigma'^2}d\sigma'+\frac{\sqrt{2\pi}l}{16}\int_{-\infty}^{\infty}\mathrm{erfc}\left(\frac{|\sigma-\sigma'|}{2\sqrt{2}l}\right)\frac{\partial^3\zeta}{\partial\sigma'^3}d\sigma'\right\}. \tag{38}$$

The effect of disorder, represented by the right-hand side, is proportional to the mean-square height of the bed roughness. Before discussing numerical solutions for finite values of D^2, we examine two limiting cases analytically.

IV. ROUGH BED WITH A SMALL MEAN-SQUARE HEIGHT

Damping of solitons by weak viscous dissipation in the bottom boundary layer has been studied before [33,34]. We follow the method in [34] (as described in [35]) and consider a case of small mean-square height $D^2 \ll O(1)$. Introducing a slow variable $X_1=D^2X$ and expand ζ into a power series of D^2,

$$\zeta = \zeta^{(0)}(X,X_1;\sigma)+D^2\zeta^{(1)}(X,X_1;\sigma)+O(D^4) \tag{39}$$

we get from (38) the perturbation equations

$$\frac{\partial\zeta^{(0)}}{\partial X}+\frac{3}{2}\zeta^{(0)}\frac{\partial\zeta^{(0)}}{\partial\sigma}+\frac{1}{6}\frac{\partial^3\zeta^{(0)}}{\partial\sigma^3} = 0, \tag{40}$$

$$\frac{\partial\zeta^{(1)}}{\partial X}+\frac{3}{2}\zeta^{(0)}\frac{\partial\zeta^{(1)}}{\partial\sigma}+\frac{3}{2}\zeta^{(1)}\frac{\partial\zeta^{(0)}}{\partial\sigma}+\frac{1}{6}\frac{\partial^3\zeta^{(1)}}{\partial\sigma^3} = -\frac{\partial\zeta^{(0)}}{\partial X_1}+\frac{1}{2}\frac{\partial\zeta^{(0)}}{\partial\sigma}+\frac{\sqrt{2\pi}l}{8}\frac{\partial^2\zeta^{(0)}}{\partial\sigma^2}+\frac{1}{16}\int_{-\infty}^{\infty}\frac{\partial^2\zeta^{(0)}}{\partial\sigma'^2}\exp\left(-\frac{(\sigma-\sigma')^2}{8l^2}\right)d\sigma'+\frac{\sqrt{2\pi}l}{16}\int_{-\infty}^{\infty}\frac{\partial^3\zeta^{(0)}}{\partial\sigma'^3}\mathrm{erfc}\left(\frac{|\sigma-\sigma'|}{2\sqrt{2}l}\right)d\sigma'\Bigg\}. \tag{41}$$

Denoting

$$\rho = \sigma - c(X_1)X \tag{42}$$

(40) can be transformed to

$$\frac{\partial}{\partial\rho}\left\{-c+\frac{3}{4}\zeta^{(0)}+\frac{1}{6}\frac{\partial^2}{\partial\rho^2}\right\}\zeta^{(0)} = 0. \tag{43}$$

The solution for $\zeta^{(0)}$ is the well-known soliton

$$\zeta^{(0)} = A\,\mathrm{sech}^2\left[\sqrt{\frac{3A}{4}}\left(\sigma-\frac{A}{2}X\right)\right], \tag{44}$$

where the local phase speed $c(X_1)=A/2$ and local wavelength $\sqrt{4/3A}$ vary slowly with X_1 through the local amplitude $A(X_1)$, whose initial value is $A(0)=1$.

Using the transformation (42), the linear equation (41) can be written as

$$\frac{\partial}{\partial \rho}\left\{-c+\frac{3}{2}\zeta^{(0)}+\frac{1}{6}\frac{\partial^2}{\partial \rho^2}\right\}\zeta^{(1)}=\mathrm{RHS}(41). \qquad (45)$$

Now the differential operators of (43) and (45) are adjoint to each other. By applying Green's identity, we obtain the solvability condition,

$$\int_{-\infty}^{\infty}\zeta^{(0)}\mathrm{RHS}(41)=\int_{-\infty}^{\infty}d\rho\zeta^{(0)}\left\{-\frac{\partial \zeta^{(0)}}{\partial X_1}+\frac{1}{2}\frac{\partial \zeta^{(0)}}{\partial \rho}+\frac{\sqrt{2\pi}l}{8}\frac{\partial^2\zeta^{(0)}}{\partial \rho^2}+\frac{1}{16}\int_{-\infty}^{\infty}\frac{\partial^2\zeta^{(0)}}{\partial \rho'^2}\times\exp\left(-\frac{(\rho-\rho')^2}{8l^2}\right)d\rho'+\frac{\sqrt{2\pi}l}{16}\int_{-\infty}^{\infty}\frac{\partial^3\zeta^{(0)}}{\partial \rho'^3}\mathrm{erfc}\left(\frac{|\rho-\rho'|}{2\sqrt{2}l}\right)d\rho'\right\}=0.$$

In the curly braces, the second and fifth term do not contribute to the integral due to their oddness in ρ. Therefore, we obtain after integration by parts,

$$-\frac{1}{2}\frac{\partial}{\partial X_1}\int_{-\infty}^{\infty}\zeta^{(0)2}d\rho-\frac{\sqrt{2\pi}l}{8}\int_{-\infty}^{\infty}\left(\frac{\partial \zeta^{(0)}}{\partial \rho}\right)^2 d\rho+\frac{1}{16}\int_{-\infty}^{\infty}d\rho\zeta^{(0)}\int_{-\infty}^{\infty}\frac{\partial^2\zeta^{(0)}}{\partial \rho'^2}e^{-[(\rho-\rho')^2/8l^2]}d\rho'=0$$

which yields finally an ordinary differential equation for the soliton amplitude $A(X_1)$

$$\frac{\partial A}{\partial X_1}=-\frac{\sqrt{2\pi}l}{10}A^2-\frac{\sqrt{3}}{16}A^{3/2}F(\sqrt{6}lA^{1/2}), \qquad (46)$$

where F is defined by

$$F(u)=2\int_0^{\infty}dp\ \mathrm{sech}^2 p\int_0^{\infty}(3\ \mathrm{sech}^4 q-2\ \mathrm{sech}^2 q)\cdot(e^{-[(q-p)^2/u^2]}+e^{-[(q+p)^2/u^2]})dq \qquad (47)$$

which is plotted in Fig. 1.

At large $X_1 \gg 1$, $A(X_1)$ becomes small. Then the second integral in (47) is dominated by contributions near $q=p$, hence $F(u)$ can be approximated for small argument by

$$F(u)\approx 2\int_0^{\infty}\mathrm{sech}^4 p(3\ \mathrm{sech}^2 p-2)dp\int_{-\infty}^{\infty}e^{-(\xi^2/u^2)}d\xi=\frac{8\sqrt{\pi}}{15}u. \qquad (48)$$

Now (46) can be approximated by

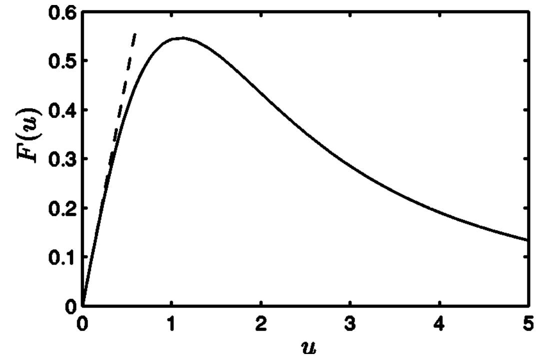

FIG. 1. Function $F(u)$ in (47). Solid line, $F(u)$; dashed line, (48).

$$\frac{\partial A}{\partial X_1}=-\frac{\sqrt{2\pi}l}{5}A^2. \qquad (49)$$

Therefore for large X_1, A decays algebraically,

$$A(X_1)\approx\frac{5}{\sqrt{2\pi}l}\frac{1}{X_1}\quad \text{as}\quad X_1 \gg 1. \qquad (50)$$

Thus, unlike sinusoidal waves in simple linear systems, a soliton, which is a transient nonlinear pulse, decays algebraically.

Subject to the initial condition $A(0)=1$, (46) can be solved numerically by standard routines. The numerical solution of (46) for different correlation length l is plotted in Fig. 2. For a random bed with short correlation length, attenuation occurs in a relatively long distance.

By employing a spectral method described in Appendix B, (38) has been solved numerically for several small values of D^2. In Fig. 3 the computed amplitude attenuation is compared with the approximate prediction here. For the smallest $D^2=0.05$, the agreement is excellent. As D^2 increases to moderate values of 0.25 and 0.5, the fully numerical solution attenuates somewhat faster at small X_1 and then slower at large X_1.

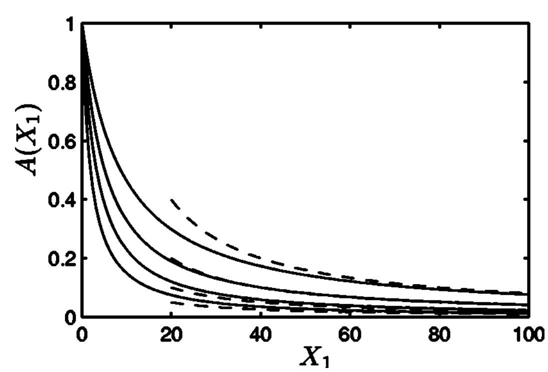

FIG. 2. Spatial decay of soliton amplitude. The solid curves correspond to a correlation length of $l=0.25, 0.5, 1.0, 2.0$ from the top down. The dashed curves are calculated from (50).

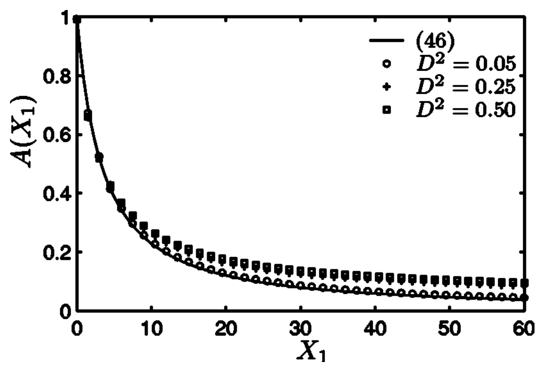

FIG. 3. Comparison of the approximate solution by (46) and numerical solution of the soliton amplitude decay, $l=1$.

V. ASYMPTOTIC BEHAVIOR FOR LONG TRAVEL DISTANCE

After traveling a long distance, the soliton is reduced and the profile flattened. Therefore, the dimensionless local characteristic wave number $K(X) \ll 1$ for large X. Because of mass conservation, the local wave amplitude $A(X)$ decays at the same rate of $K(X)$. In (38), the diffusion and dispersion terms can be compared to the inertia term as follows:

$$\frac{\partial^3\zeta}{\partial\sigma^3}\bigg/\zeta\frac{\partial\zeta}{\partial\sigma} \sim \frac{K^2(X)}{A(X)} \sim O(K(X)) \ll 1,$$

$$\frac{\partial^2\zeta}{\partial\sigma^2}\bigg/\zeta\frac{\partial\zeta}{\partial\sigma} \sim \frac{K(X)}{A(X)} \sim O(1),$$

i.e., dispersion is negligible compared with nonlinearity and dissipation. Since at large X the length scale of ζ is much greater than the initial soliton length which is comparable to the correlation length l of disorder, the third term on the right-hand side of (38) can be approximated by

$$\frac{1}{16}\int_{-\infty}^{\infty}\exp\left(-\frac{|\sigma-\sigma'|^2}{8l^2}\right)\frac{\partial^2\zeta}{\partial\sigma'^2}d\sigma' \approx \frac{1}{16}\frac{\partial^2\zeta}{\partial\sigma^2}\int_{-\infty}^{\infty}\times\exp\left(-\frac{\sigma'^2}{8l^2}\right)d\sigma' = \frac{\sqrt{2\pi}l}{8}\frac{\partial^2\zeta}{\partial\sigma^2}. \tag{51}$$

Hence, the asymptotic form of (38) for $X \gg 1$ reduces to

$$\frac{\partial\zeta}{\partial X}+\frac{3}{2}\zeta\frac{\partial\zeta}{\partial\sigma}=\frac{D^2}{2}\frac{\partial\zeta}{\partial\sigma}+\frac{\sqrt{2\pi}D^2l}{4}\frac{\partial^2\zeta}{\partial\sigma^2}. \tag{52}$$

We now change to a new reference frame moving at the speed $-D^2/2$ with respect to the σ system, and introduce

$$\rho=\sigma+\frac{D^2X}{2}. \tag{53}$$

Equation (52) becomes Burgers' equation

$$\frac{\partial\zeta}{\partial X}+\frac{3}{2}\zeta\frac{\partial\zeta}{\partial\rho}=\frac{\sqrt{2\pi}D^2l}{4}\frac{\partial^2\zeta}{\partial\rho^2} \tag{54}$$

for which an initial-value problem has been solved analytically by Cole and Hopf. It can be expected that at large X, the asymptotic profile of our soliton should approach that of the Cole-Hopf solution for the initial data $\zeta(\rho,0)=(4/\sqrt{3})\delta(\rho)$ which has the same area of $4/\sqrt{3}$ as our initial soliton (see (44) with $A=1$), namely,

$$\zeta \approx \left(\frac{\sqrt{2\pi}D^2l}{X}\right)^{1/2}\frac{1}{3}\mathcal{F}(\rho'), \quad X \gg 1 \tag{55}$$

(see [36]), where

$$\mathcal{F}(\rho')=\frac{N}{\sqrt{\pi}}\frac{e^{-\rho'^2}}{1+\frac{N}{2}\text{erfc }\rho'} \tag{56}$$

with

$$\rho'=\frac{\rho}{(\sqrt{2\pi}D^2lX)^{1/2}} \tag{57}$$

and

$$N=\exp\left(\sqrt{\frac{24}{\pi}}\frac{1}{D^2l}\right)-1, \tag{58}$$

for any degree of disorder D^2l. Thus for large X, the asymptotic profile of (55) is a bell-shaped crest with a steeper front moving in the coordinate σ at the negative speed $D^2/2$. The crest attenuates in height as $1/\sqrt{X}$ and expands in width as $\sqrt{X}$, so that the total area is conserved. It is easy to see that the total energy in the pulse decays at the rate of $1/\sqrt{X}$.

For large D^2l, i.e., strong disorder,

$$N \approx \sqrt{\frac{24}{\pi}}\frac{1}{D^2l} \ll 1 \tag{59}$$

and (55) can be approximated by

$$\zeta \approx \frac{4}{\sqrt{3\pi}}\left(\frac{1}{\sqrt{2\pi}D^2lX}\right)^{1/2}e^{-(\rho^2/\sqrt{2\pi}D^2lX)}. \tag{60}$$

Nonlinearity is overwhelmed by diffusion.

Let us define the normalized attenuation length L to be the distance over which the maximum wave height decreases to some small fraction (say, 1/10) of its initial value

$$A(X=L)=\frac{1}{10}A(X=0)=\frac{1}{10}. \tag{61}$$

For small D^2l, L can be estimated by (50), i.e.,

$$L \approx \frac{50}{\sqrt{2\pi}D^2l}. \tag{62}$$

For intermediate $D^2l=O(1)$, we use (58) to get

$$\frac{1}{10}=\frac{1}{3}\left(\frac{\sqrt{2\pi}D^2l}{L}\right)^{1/2}\max(\mathcal{F}). \tag{63}$$

It is straightforward to find $\max(\mathcal{F})=2\rho_0'$ which occurs when $\rho'=\rho_0'$ is the solution of the transcendental equation

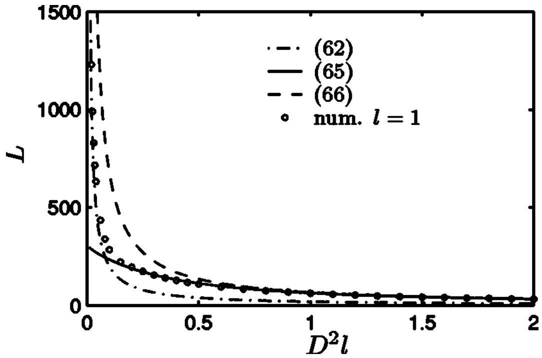

FIG. 4. Attenuation length by approximate formulas and numerical computation.

$$\rho'(2 + N\ \mathrm{erfc}\ \rho') = \frac{N}{\sqrt{\pi}} e^{-\rho'^2} \tag{64}$$

and depends on D^2l through (58). It then follows that

$$L \approx \frac{400\sqrt{2\pi}D^2l}{9}\rho_0'^2. \tag{65}$$

Equation (65) holds also for large D^2l. A more explicit expression can be had by using the limit (60),

$$L \approx \frac{1600}{3\pi\sqrt{2\pi}D^2l}. \tag{66}$$

In Fig. 4 the attenuation length is shown by three approximations according to (62), (65), and (66) for different range of D^2l. These results agree well with the direct numerical computations as described in the next section. Clearly higher roughness and/or larger correlation length lead to faster attenuation.

We now discuss numerical solutions to the nonlinear integro-differential equation (38) for two initial-value problems obtained by a spectral method described in Appendix B.

VI. TRANSFORMATION OF A SOLITON OVER A LONG ROUGH SEABED

We first discuss the evolution of a soliton over the total distance of $0 \leqslant X \leqslant 100$ covered by roughness of given D and l. Two seabeds with low-amplitude roughnesses are first chosen for the same correlation length of $l=1$. The numerical solutions are shown in Figs. 5(a) and 5(b). As X increases, the wave profiles flatten gradually with X. For the lower roughness, the wave crest first travels forward in the moving coordinate, therefore faster than the linear phase speed over a smooth bed (1 in dimensionless form or $\sqrt{gH}$ in physical dimensions, in the stationary frame of reference). As the crest loses its height with X, it also slows down to below the linear phase speed. By comparing Figs. 5(a) and 5(b), the soliton is slowed down sooner by the bed with higher roughness. For large X, the asymptotic profile is described by (55).

With still higher roughness, solitons are further slowed, as shown in Figs. 5(c) and 5(d). For $D^2=0.5$, the forward push by inertia loses more ground to retardation by roughness, see

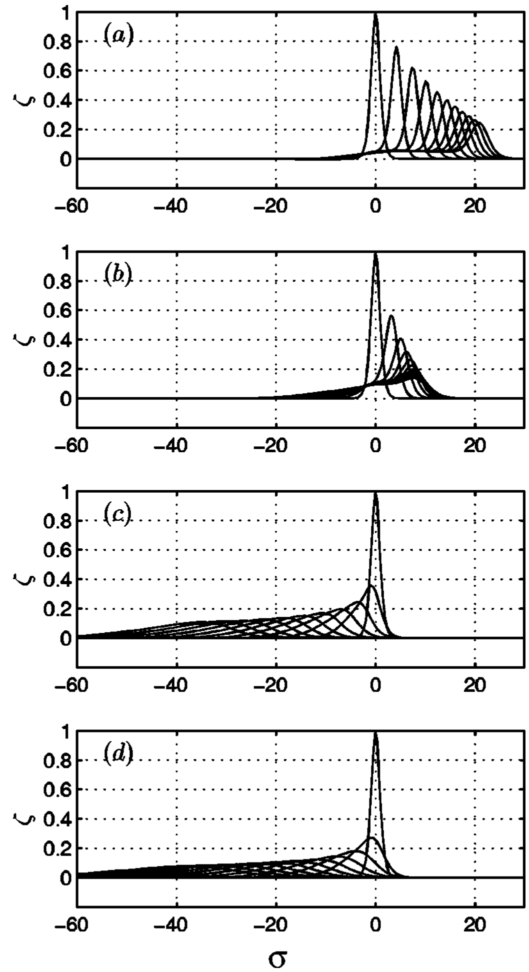

FIG. 5. Effects of roughness amplitude on soliton evolution over a random seabed. The total travel distance is $100(0 \leqslant X \leqslant 100)$. Wave profiles are shown at every $\Delta X=10$. $l=1$, (a). $D^2=0.10$, (b). $D^2=0.25$, (c). $D^2=0.50$; (d). $D^2=1.0$.

Fig. 5(c). For the highest roughness with $D^2=1.0$ [see Fig. 5(d)], inertia is overpowered by roughness and nonlinearity is no longer effective. The wave crest always travels at a speed lower than the linear wave speed for a smooth bed.

As a larger l corresponds also to stronger disorder, stronger dissipation and faster attenuation are expected, and are confirmed by Figs. 6(a) and 6(b) for the same $D^2=0.1$. If the seabed roughness is higher, the effect of correlation length is more important, as seen in Figs. 7(a) and 7(b).

VII. SOLITON FISSION AFTER PASSING A RANDOM STRIP

Across a linear random medium, a sinusoidal wave train loses its amplitude. After crossing a finite strip of disorder, the transmitted wave diminishes exponentially with the strip width, but remains sinusoidal. For the nonlinear dispersive system here, it is interesting to see whether the much flattened, transmitted pulse, which is no longer a soliton, may lead to fission of new solitons.

Let us consider the evolution of a transient pulse, after it emerges from a random strip of length X_0. The computed

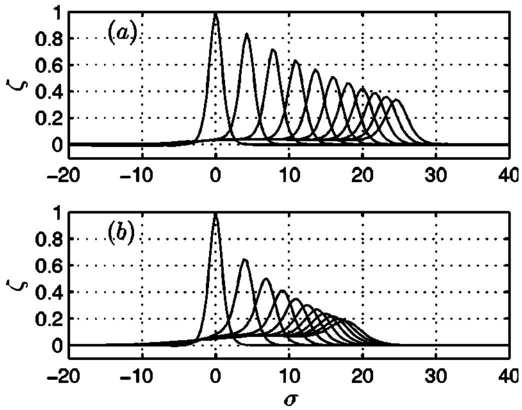

FIG. 6. Effects of correlation length on soliton evolution over a random seabed. The total distance of travel is 100 $(0 \leqslant X \leqslant 100)$. Wave profiles are shown at every $\Delta X = 10$. $D^2 = 0.10$, (a) $l = 0.5$; (b) $l = 2.0$.

$\zeta(\sigma, X_0)$ is used as the initial data to compute the subsequent evolution of $\zeta(\sigma, X)$ for $X > X_0$ from the classical KdV equation. By dropping the right-hand side of (38) and using the transformation,

$$\zeta \to -\overline{\zeta}, \quad \sigma \to \left(\frac{2}{3}\right)^{1/2} \overline{\sigma} X - X_0 \to 6\left(\frac{2}{3}\right)^{3/2} \tau \qquad (67)$$

we reduce (38) with $D^2 = 0$ to the canonical form

$$\overline{\zeta}_\tau - 6\overline{\zeta}\,\overline{\zeta}_{\overline{\sigma}} + \overline{\zeta}_{\overline{\sigma}\overline{\sigma}\overline{\sigma}} = 0. \qquad (68)$$

According to the inverse scattering theory of Gardner *et al.* [37], an initial profile $\overline{\zeta}(\overline{\sigma}, 0)$ can disintegrate into a number of new solitons if real negative eigenvalues can be found from the following Schrödinger equation with $\overline{\zeta}(\overline{\sigma}, 0)$ as the potential:

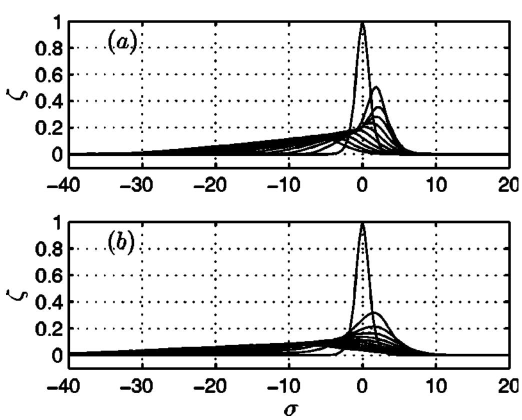

FIG. 7. Effects of correlation length on soliton evolution over a random seabed. The total distance of travel is 100 $(0 \leqslant X \leqslant 100)$. Wave profiles are shown at every $\Delta X = 10$. $D^2 = 0.50$, (a) $l = 0.5$; (b) $l = 2.0$.

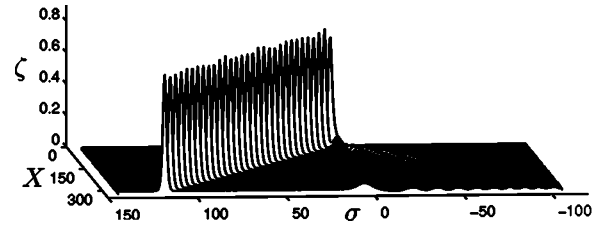

FIG. 8. Fission of solitonlike pulses after a soliton passes a finite strip of random seabed. $l = 1.0$, $X_0 = 5$, $D^2 = 0.25$. The last profile is at $X - X_0 = 300$.

$$-\frac{\partial^2 \psi}{\partial x^2} + \overline{\zeta}(\overline{\sigma}, 0)\psi = \lambda \psi. \qquad (69)$$

The number of solitons is equal to the number of eigenvalues λ, roughly given by $\sqrt{A_0}L$, where A_0 is the height and L the width of the initial pulse $\overline{\zeta}(\overline{\sigma}, 0)$.

For the fixed correlation length $l = 1$, we have first computed the soliton evolution over rough beds of different mean square heights D^2 and widths X_0. For each set of (D, X_0) the final profile at X_0 is used as the initial data to compute the subsequent fission by solving the usual KdV equation (40). Typical long-time evolutions are shown in Figs. 8–11. Fission into two, three, four, and five separate pulses, after a soliton escapes a finite random bed with the total extent $X_0 = 5$, 20, 30, and 50, can be seen. The tallest pulses are essentially solitons.

From many numerical solutions of the eigenvalue problems governed by (69), the number of disintegrated solitonlike pulses is displayed in the plane of D^2 and X_0, all for $l = 1$. The thresholds of fission are found to be hyperbolas $D^2 X_0 = \text{constant}$, as shown in Fig. 12. Thus more solitons of diminishing amplitudes emerge after transmission if either D^2 or X_0 is larger.

We first examine why the number of solitons increases monotonically with $D^2 X_0$. Recall from Sec. V that after traveling a distance X_0 over a random bed a soliton flattens by diffusion to a pulse of height proportional to $(D^2 l X_0)^{-1/2}$ and length proportional to $(D^2 l X_0)^{1/2}$. Their product, which determines the number of eigenvalues and the number of solitons in the inverse scattering theory, is

$$(\text{height}) \times (\text{length})^2 \propto (D^2 l X_0)^{1/2}. \qquad (70)$$

Hence it is the flattening effect of diffusion that leads to the proliferation of solitons with increasing $D^2 l X_0$, as found numerically.

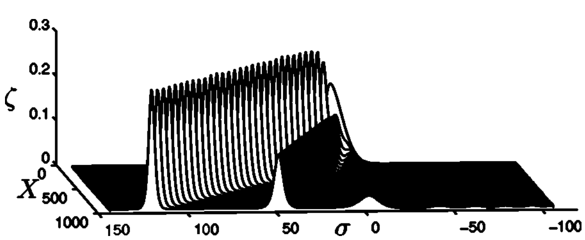

FIG. 9. Fission of solitonlike pulses after a soliton passes a finite strip of random seabed. $l = 1.0$, $X_0 = 20$, $D^2 = 1.0$. The last profile is at $X - X_0 = 900$.

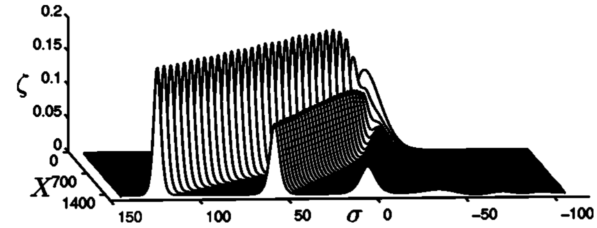

FIG. 10. Fission of solitonlike pulses after a soliton passes a finite strip of random seabed. $l=1.0$, $X_0=30$, $D^2=1.5$. The last profile is at $X-X_0=1400$.

Next, we point out that the proliferation of vanishingly small solitons is of little dynamical consequence since they are geometrical forms of negligible energy. As seen from (44), the classical soliton of any dimensionless amplitude A_0, has the dimensionless wavelength $1/K$ where $K=\sqrt{3A_0/4}$. Although the corresponding ratio of nonlinearity to dispersion (Ursell parameter) $A_0/K^2=4/3$ is independent of A_0, the total energy of a soliton is proportional to

$$\int_{-\infty}^{\infty} \zeta^2 d\sigma = A_0^2 \int_{-\infty}^{\infty} \operatorname{sech}^4 \sqrt{\frac{3A_0}{4}}\sigma d\sigma = \left(\frac{3A_0}{4}\right)^{3/2}. \quad (71)$$

Therefore, a soliton of vanishingly small amplitude has vanishingly small energy. In conclusion, past a longer or more disordered rough bed, a soliton becomes a flatter pulse after transmission, and procreates a larger number of solitons of vanishingly small amplitude and energy.

VIII. CONCLUDING REMARKS

For long waves over a randomly rough seabed, we have studied the accumulated effects of multiple scattering by disordered irregularities on the seabed. In addition to the usual assumptions of KdV approximation that nonlinearity is comparable to dispersion, we assume that the ratio of randomness height to mean depth is comparable to that of mean depth to the characteristic wavelength. Disorder is shown to cause diffusion, which leads to spatial attenuation of amplitude, flattening of profile, slowing down of wave advance, and reduction of dispersion. After a large region of random scattering, dispersion loses ground to diffusion, while nonlinearity can still remain. Unlike sinusoidal waves in linear cases, attenuation with distance is algebraic rather than exponential. If the random region is finite in length, the transmitted wave is a flattened pulse which disintegrates into many small and flat solitons of vanishing energy.

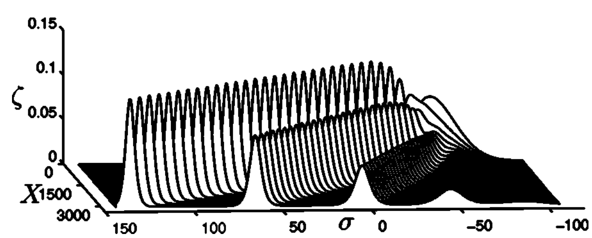

FIG. 11. Fission of solitonlike pulses after a soliton passes a finite strip of random seabed. $l=1.0$, $X_0=50$, $D^2=2.5$. The last profile is at $X-X_0=3000$.

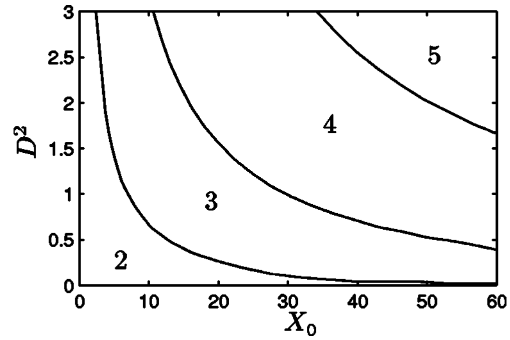

FIG. 12. Number of disintegrating solitons after a finite random seabed. $l=1.0$.

Similar analyses of problems in geophysical fluid dynamics such as internal waves, and in fiber optics, should be of considerable scientific and engineering interest.

ACKNOWLEDGMENTS

We acknowledge with gratitude the financial support by the U.S. Office of Naval Research (Grant No. N00014-89J-3128, Dr. Thomas Swean) and the U.S. National Science Foundation (Grant No. CTS-0075713, Dr. John Foss, Dr. C. F. Chen, and Dr. Michael W. Plesniak).

APPENDIX A: DETAILS OF RANDOM FORCING

In terms of the Fourier transform, we can write

$$\frac{\partial \zeta}{\partial x'} = \frac{1}{2\pi}\int_{-\infty}^{\infty} ike^{ik(x'-x-(t'-t))}\hat{\zeta}(k,X)e^{ik(x-t)}dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} ike^{ik\xi}e^{-ik\tau}\hat{\zeta}(k,X)e^{ik(x-t)}dk, \quad (A1)$$

where

$$\xi = x' - x, \quad \tau = t' - t.$$

Note that

$$\frac{\partial G}{\partial x} = \frac{\partial G}{\partial |\xi|}\operatorname{sgn}(\xi) = -\frac{1}{2}\delta(\tau - |\xi|)\operatorname{sgn}(\xi). \quad (A2)$$

Using these results in (18), we get, after simple variable transformations, that

$$-\left\langle b(x)\frac{\partial \eta_1}{\partial x}\right\rangle = \frac{1}{4\pi}\int\int\int_{-\infty}^{\infty} d\tau d\xi dk e^{ik(x-t)} ik\frac{\partial}{\partial \xi}$$

$$\times[\Gamma(\xi)\hat{\zeta}(k)e^{-ik\xi}]e^{ik\tau}\delta(\tau - |\xi|)\operatorname{sgn}(\xi).$$

Since

$$\int_{-\infty}^{\infty} d\tau e^{ik\tau}\delta(\tau - |\xi|) = e^{ik|\xi|}$$

the triple integral above becomes

$$-\left\langle b(x)\frac{\partial \eta_1}{\partial x}\right\rangle = \frac{1}{2\pi}\int\int_{-\infty}^{\infty} d\xi dk e^{ik(x-t)}\hat{\zeta}(k)\frac{ik}{2}\frac{\partial}{\partial \xi}\times[\Gamma(\xi,X)e^{-ik\xi}]e^{ik|\xi|}\,\mathrm{sgn}(\xi),$$

which gives (20) after using the definition (21).

APPENDIX B: NUMERICAL METHOD FOR SOLVING EQ. (38)

We choose a computational domain $[0<\sigma<L]$ and impose periodic boundary conditions at both ends. The domain must be large enough so that there no wave disturbance reaches either end during computation. We first scale the spacelike variable σ so that the computational domain is 2π

$$\xi = K_L\sigma = \frac{2\pi}{L}\sigma. \tag{B1}$$

The governing equation (38) is rewritten as

$$\frac{\partial \zeta}{\partial X} + \frac{3K_L}{2}\zeta\frac{\partial \zeta}{\partial \xi} + \frac{K_L^3}{6}\frac{\partial^3\zeta}{\partial \xi^3} = D^2\left\{\frac{K_L}{2}\frac{\partial \zeta}{\partial \xi} + \frac{\sqrt{2\pi}lK_L^2}{8}\frac{\partial^2\zeta}{\partial \xi^2} + \frac{K_L}{16}\int_{-\infty}^{\infty}\frac{\partial^2\zeta}{\partial \xi'^2}\times\exp\left(-\frac{(\xi-\xi')^2}{8K_L^2l^2}\right)d\xi' + \frac{\sqrt{2\pi}K_L^2 l}{16}\int_{-\infty}^{\infty}\frac{\partial^3\zeta}{\partial \xi'^3}\times\mathrm{erfc}\left(\frac{|\xi-\xi'|}{2\sqrt{2}K_L l}\right)d\xi'\right\}. \tag{B2}$$

The displacement ζ at uniformly spaced nodes is then represented by a discrete Fourier series

$$\zeta(\xi_j) = \sum_{m=-N/2+1}^{N/2}\hat{\zeta}_m e^{im\xi_j}, \tag{B3}$$

where $\xi_j = 2\pi j/N$.

The discrete Fourier transform of (B2) is

$$\frac{\partial \hat{\zeta}_m}{\partial X} = \mathcal{G}(mK_L)\hat{\zeta}_m - \frac{3imK_L}{4}\mathcal{F}_m(\zeta^2), \tag{B4}$$

where $\mathcal{F}_m$ denotes the mth component of discrete Fourier transform and

$$\mathcal{G}(k) = -\frac{\sqrt{2\pi}}{8}k^2D^2l(1+e^{-2k^2l^2}) + ik\frac{D^2}{2} + ik^3\left(\frac{1}{6} - \frac{D^2l^2}{2}\frac{\sqrt{\pi}e^{-2k^2l^2}\,\mathrm{erfi}(\sqrt{2}kl)}{2\sqrt{2}kl}\right). \tag{B5}$$

The fourth order Runge-Kutta method is used to solve (B4) numerically:

$$\hat{\zeta}_m^{n+1} = \hat{\zeta}_m^n + \frac{\Delta t}{6}(\hat{\zeta}_{k1,m} + 2\hat{\zeta}_{k2,m} + 2\hat{\zeta}_{k3,m} + \hat{\zeta}_{k4,m}), \tag{B6}$$

where

$$\hat{\zeta}_{k1,m} = \mathcal{G}(mK_L)\hat{\zeta}_m^n - \frac{3imK_L}{4}\mathcal{F}_m((\zeta^n)^2), \tag{B7}$$

$$\hat{\zeta}_{k2,m} = \mathcal{G}(mK_L)\left(\hat{\zeta}_m^n + \frac{\Delta t}{2}\hat{\zeta}_{k1,m}\right) - \frac{3imK_L}{4}\mathcal{F}_m\left(\left(\zeta + \frac{\Delta t}{2}\zeta_{k1}\right)^2\right), \tag{B8}$$

$$\hat{\zeta}_{k3,m} = \mathcal{G}(mK_L)\left(\hat{\zeta}_m^n + \frac{\Delta t}{2}\hat{\zeta}_{k2,m}\right) - \frac{3imK_L}{4}\mathcal{F}_m\left(\left(\zeta + \frac{\Delta t}{2}\zeta_{k2}\right)^2\right), \tag{B9}$$

$$\hat{\zeta}_{k4,m} = \mathcal{G}(mK_L)(\hat{\zeta}_m^n + \Delta t\hat{\zeta}_{k3,m}) - \frac{3imK_L}{4}\mathcal{F}_m((\zeta + \Delta t\zeta_{k3})^2). \tag{B10}$$

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